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RAGHUNATHAN'S CONJECTURES FOR SL(2, R)

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ABSTRACT

In this paper I give simple proofs of Raghunathan's conjectures for SL(2, R). These proofs incorporate in a simplified form some of the ideas and methods I used to prove the Raghunathan's conjectures for general connected Lie groups.

Introduction

The purpose of this paper is to present simple proofs of Raghunathan's conjectures for SL(2, R).

More specifically, let $\mathbf{G}$ be a Lie group with the Lie algebra $\mathfrak{G}$, Γ a discrete subgroup of $\mathbf{G}$ and $\pi : \mathbf{G} \to \Gamma\backslash\mathbf{G}$ the covering projection $\pi(\mathbf{g}) = \Gamma\mathbf{g}$, $\mathbf{g} \in \mathbf{G}$. The group $\mathbf{G}$ acts by right translations on $\Gamma\backslash\mathbf{G}$, $x \to x\mathbf{g}$, $x \in \Gamma\backslash\mathbf{G}$, $\mathbf{g} \in \mathbf{G}$. A subset $A \subset \Gamma\backslash\mathbf{G}$ is called **homogeneous** if there is $\mathbf{x} \in \mathbf{G}$ and a closed subgroup $\mathbf{H} \subset \mathbf{G}$ such that $\mathbf{xHx}^{-1} \cap \Gamma$ is a lattice in $\mathbf{xHx}^{-1}$ and $A = \pi(\mathbf{x})\mathbf{H}$. A Borel probability measure μ on $\Gamma\backslash\mathbf{G}$ is called **algebraic** if there exists $x \in \Gamma\backslash\mathbf{G}$ and a closed subgroup $\mathbf{H} \subset \mathbf{G}$ such that $x\mathbf{H}$ is homogeneous and μ is the $\mathbf{H}$-invariant Borel probability measure supported on $x\mathbf{H}$.

A subgroup $\mathbf{U} \subset \mathbf{G}$ is called **unipotent** if for each $\mathbf{u} \in \mathbf{U}$ the map $\mathrm{Ad}_{\mathbf{u}} : \mathfrak{G} \to \mathfrak{G}$ is a unipotent linear transformation of $\mathfrak{G}$.

Here are the two Raghunathan's conjectures.

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CONJECTURE 1 (Raghunathan's Topological Conjecture): *Let* $\mathbf{G}$ *be a connected Lie group and* $\mathbf{U}$ *a unipotent subgroup of* $\mathbf{G}$. *Then given any lattice* Γ *in* $\mathbf{G}$ *and any* $x \in \Gamma\backslash\mathbf{G}$, *the closure* $\overline{x\mathbf{U}}$ *of the orbit* $x\mathbf{U}$ *in* $\Gamma\backslash\mathbf{G}$ *is homogeneous.*

CONJECTURE 2 (Raghunathan's Measure Conjecture): *Let* $\mathbf{G}$ *be a connected Lie group and* $\mathbf{U}$ *a unipotent subgroup of* $\mathbf{G}$. *Then given any lattice* Γ *of* $\mathbf{G}$, *every ergodic* $\mathbf{U}$*-invariant Borel probability measure on* $\Gamma\backslash\mathbf{G}$ *is algebraic.*

In fact, Raghunathan proposed a weaker version of Conjecture 1. This version and Conjecture 2 were stated by Dani [D1] for reductive $\mathbf{G}$ and by Margulis [M1, Conjectures 2 and 3] for general $\mathbf{G}$.

Conjectures 1 and 2 for nilpotent $\mathbf{G}$ were proved earlier by Parry [P] and Furstenberg [F1] and for $\mathbf{G} = \mathrm{SL}(2, R)$ by Hedlund [H], Furstenberg [F2] and Dani [D1].

Recently Conjecture 1 and a stronger verison of Conjecture 2 were proved in [R1-4]. More specifically, we proved the following theorems.

THEOREM A (Orbit closures for unipotent actions): *Let* $\mathbf{G}$ *be a connected Lie group and* $\mathbf{U}$ *a unipotent subgroup of* $\mathbf{G}$. *Then given any lattice* Γ *of* $\mathbf{G}$ *and any* $x \in \Gamma\backslash\mathbf{G}$ *the closure* $\overline{x\mathbf{U}}$ *of the orbit* $x\mathbf{U}$ *in* $\Gamma\backslash\mathbf{G}$ *is homogeneous.*

THEOREM B (Classification of invariant measures for unipotent actions): *Let* $\mathbf{G}$ *be a connected Lie group and* $\mathbf{U}$ *a unipotent subgroup of* $\mathbf{G}$. *Then given* **any** *discrete subgroup* Γ *(not necessarily a lattice) of* $\mathbf{G}$, *every ergodic* $\mathbf{U}$*-invariant Borel probability measure on* $\Gamma\backslash\mathbf{G}$ *is algebraic.*

Now let $\mathbf{U} = \{\mathbf{u}(t) = \exp tu \colon t \in R\}$, $u \in \mathfrak{G}$ be a one-parameter subgroup of $\mathbf{G}$. A point $x \in \Gamma\backslash\mathbf{G}$ is called **generic** for $\mathbf{U}$ if there exists a closed subgroup $\mathbf{H} \subset \mathbf{G}$ such that $\mathbf{U} \subset \mathbf{H}$, $\overline{x\mathbf{U}} = x\mathbf{H}$ is homogeneous and $\frac{1}{t}\int_0^t f(x\mathbf{u}(s))ds \to \int_{\Gamma\backslash\mathbf{G}} f d\nu_{\mathbf{H}}$ for every bounded continuous function f on $\Gamma\backslash\mathbf{G}$, where $\nu_{\mathbf{H}}$ denotes the $\mathbf{H}$-invariant Borel probability measure on $\Gamma\backslash\mathbf{G}$, supported on $x\mathbf{H}$. Similarly, one defines generic points for one-generator subgroups $\mathbf{U} = \{\mathbf{u}^k \colon k \in \mathbb{Z}\}$ of $\mathbf{G}$, $\mathbf{u} \in \mathbf{G}$.

In [R4] we proved the following theorem.

THEOREM C (Uniform distribution of unipotent orbits): *Let* $\mathbf{G}$ *be a connected Lie group,* Γ *a lattice in* $\mathbf{G}$ *and* $\mathbf{U}$ *a one-parameter or one-generator unipotent subgroup of* $\mathbf{G}$. *Then every point* $x \in \Gamma\backslash\mathbf{G}$ *is generic for* $\mathbf{U}$.

Theorem C was conjectured by Margulis in [M2, Conjectures 3 and 4]. For $\mathbf{G} = \mathrm{SL}(2, R)$ Theorem C was proved by Dani and Smillie in [DS]. Also recently

N. Shah [Sh] proved Theorem C for semisimple $\mathbf{G}$ of real rank one by other methods.

We conjecture the following version of Theorem C for arbitrary Γ (not necessarily lattices).

CONJECTURE D: *Let* $\mathbf{G}$ *be a connected Lie group,* Γ *a discrete subgroup of* $\mathbf{G}$ *and* $\mathbf{U}$ *a unipotent subgroup of* $\mathbf{G}$. *Suppose that* $x \in \Gamma\backslash\mathbf{G}$ *and* $\overline{x\mathbf{U}}$ *is compact in* $\Gamma\backslash\mathbf{G}$. *Then* 1) $\overline{x\mathbf{U}}$ *is homogeneous;* 2) *if* $\mathbf{U}$ *is a one-parameter or one-generator subgroup of* $\mathbf{G}$ *then* x *is generic for* $\mathbf{U}$.

The purpose of this paper is to take the simplest case of $\mathbf{G} = \mathrm{SL}(2, R)$ and to demonstrate in a simplified form some of the ideas and techniques we use to prove Theorems A, B and C. For $\mathbf{G} = \mathrm{SL}(2, R)$ we consider

$$\mathbf{U} = \left\{\mathbf{u}(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} : t \in R\right\} \quad \text{and} \quad \mathbf{A} = \left\{\begin{bmatrix} e^t & \\ & e^{-t} \end{bmatrix} : t \in R\right\}.$$

The action of $\mathbf{U}$ on $\Gamma\backslash\mathbf{G}$ is called the horocycle flow and the action of $\mathbf{A}$ on $\Gamma\backslash\mathbf{G}$ the geodesic flow. Theorems A, B, C and Conjecture D for $\mathbf{G} = \mathrm{SL}(2, R)$ take the following form.

THEOREM 1 (Orbit closures for horocycle flows): *Let* Γ *be a lattice in* $\mathbf{G} = \mathrm{SL}(2, R)$ *and* $x \in \Gamma\backslash\mathbf{G}$. *Then either* $\overline{x\mathbf{U}} = \Gamma\backslash\mathbf{G}$ *or the orbit* $x\mathbf{U} = \overline{x\mathbf{U}}$ *is periodic.*

THEOREM 2 (Classification of invariant measures for horocycle flows): *Let* Γ *be a discrete subgroup of* $\mathbf{G} = \mathrm{SL}(2, R)$ *and* μ *an ergodic* $\mathbf{U}$*-invariant Borel probability measure on* $\Gamma\backslash\mathbf{G}$. *Then either* 1) Γ *is a lattice and* μ *is* $\mathbf{G}$*-invariant or* 2) μ *is supported on a periodic orbit of* $\mathbf{U}$.

THEOREM 3 (Uniform distribution of horocycle orbits): *Let* Γ *be a lattice in* $\mathbf{G} = \mathrm{SL}(2, R)$. *Then every point* $x \in \Gamma\backslash\mathbf{G}$ *is generic for* $\mathbf{U}$. *Equivalently, if* $x \in \Gamma\backslash\mathbf{G}$ *and* $x\mathbf{U}$ *is not a periodic orbit, then* $\frac{1}{t}\int_0^t f(x\mathbf{u}(s))ds \to \int_{\Gamma\backslash\mathbf{G}} f d\nu_{\mathbf{G}}$ *for every bounded continuous function* f *on* $\Gamma\backslash\mathbf{G}$, *where* $\nu_{\mathbf{G}}$ *denotes the* $\mathbf{G}$*-invariant Borel probability measure on* $\Gamma\backslash\mathbf{G}$.

THEOREM 4: *Let* Γ *be a discrete subgroup of* $\mathbf{G} = \mathrm{SL}(2, R)$, *which is not a lattice. Suppose that* $x \in \Gamma\backslash\mathbf{G}$ *and* $\overline{x\mathbf{U}}$ *is compact in* $\Gamma\backslash\mathbf{G}$. *Then* $x\mathbf{U} = \overline{x\mathbf{U}}$ *is a periodic orbit.*

Also we include the following theorem, proved earlier in [Sa] by other methods.

THEOREM 5 (Equidistribution of closed horocycles): *Let* Γ *be a nonuniform lattice in* $\mathbf{G} = \mathrm{SL}(2, R)$ *and let* $P = \{x \in \Gamma\backslash\mathbf{G}\colon x\mathbf{U}$ *is a periodic orbit*$\}$. *Then*

$$\lim_{T(x)\to\infty} \frac{1}{T(x)} \int_0^{T(x)} f(x\mathbf{u}(s))ds = \int_{\Gamma\backslash\mathbf{G}} f d\nu_{\mathbf{G}}$$

for every bounded continuous function f *on* $\Gamma\backslash\mathbf{G}$, *where* $x \in P$ *and* $T(x) > 0$ *denote the period of the periodic orbit* $x\mathbf{U}$.

The paper is organized as follows. In section 2 we give short and rather elementary proofs of Theorem 2 for lattices, Theorem 1, Theorem 5 and Theorem 3. These proofs use in an essential way a special feature of $\mathbf{U}$ called "horosphericity" of $\mathbf{U}$ with respect to $\mathbf{A}$. This feature is not necessarily possessed by unipotent $\mathbf{U}$ in general $\mathbf{G}$. Because of this, the proofs in section 2 can not be extended to general $\mathbf{G}$. This obstacle is removed in sections 3 and 4, where we give different yet still simple proofs of Theorems 3 and 2. Moreover, section 4 handles the case of **arbitrary** discrete Γ (not necessarily lattices). The proofs in sections 3 and 4 incorporate in a simple form some of the ideas and techniques used to prove Theorems A, B and C in [R1-4]. Also we prove Theorem 4 in section 4. The argument in the proof of this theorem can be used to prove Conjecture D for semisimple $\mathbf{G}$ of real rank one. Sections 3 and 4 can be read independently of section 2 and section 4 independently of section 3. We note that all our proofs are totally different from the proofs obtained by other authors.

Finally, we point out a profound contrast in the dynamical behavior of the horocycle and the geodesic flows on $\Gamma\backslash\mathrm{SL}(2, R)$. It was shown by Sinai [S] and Bowen, Ruelle [BR] that there are infinitely many ergodic $\mathbf{A}$-invariant Borel probability measures all supported on $\Gamma\backslash\mathbf{G}$, which are not algebraic. Also there exist points $x \in \Gamma\backslash\mathbf{G}$ for which the closures $\overline{x\mathbf{A}}$ of geodesic orbits are not smooth manifolds. These facts put geodesic actions in a striking contrast with the rigid behavior of horocycle actions, given in Theorems 1, 2 and 3.

1. Preliminaries

Henceforth unless otherwise stated we shall denote by $\mathbf{G}$ the group $\mathrm{SL}(2, R)$ of all 2×2 real matrices with determinant 1, equipped with a left invariant Riemannian metric. There are the following basic one-parameter subgroups of $\mathbf{G}$:

$$\mathbf{U} = \left\{ \mathbf{u}(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} : t \in R \right\},$$
$$\mathbf{A} = \left\{ \mathbf{a}(t) = \begin{bmatrix} e^t & \\ & e^{-t} \end{bmatrix} : t \in R \right\},$$
$$\mathbf{H} = \left\{ \mathbf{h}(t) = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} : t \in R \right\}.$$

These subgroups of $\mathbf{G}$ satisfy the following commutation relations

$$\begin{aligned} &\mathbf{u}(s)\mathbf{a}(t) = \mathbf{a}(t)\mathbf{u}(se^{-2t}), \\ &\mathbf{h}(s)\mathbf{a}(t) = \mathbf{a}(t)\mathbf{h}(se^{2t}), \quad s, t \in R. \end{aligned} \tag{1.1}$$

Let $\mathbf{W}$ denote the subgroup of $\mathbf{G}$ generated by $\mathbf{A}$ and $\mathbf{H}$. For $\mathbf{x} \in \mathbf{G}$, $\delta > 0$ define $\mathbf{W}(\mathbf{x};\delta) = \{\mathbf{x}\mathbf{a}(\tau)\mathbf{h}(b)\colon |\tau| < \delta, |b| < \delta\}$. It is a fact that if $\delta > 0$ is sufficiently small then for each $\mathbf{y} \in \mathbf{W}(\mathbf{x};\delta)$ and each $0 \le s \le 1$ there is a unique $\alpha(\mathbf{y}, s) > 0$, $\alpha(\mathbf{y}, 0) = 0$ increasing in s and continuous in $(\mathbf{y}, s)$ such that

$$\psi_s(\mathbf{y}) = \mathbf{y}\mathbf{u}(\alpha(\mathbf{y}, s)) \in \mathbf{W}(\mathbf{x}\mathbf{u}(s); 10\delta).$$

The map ψ_s, $0 \le s \le 1$ is a homeomorphism from $\mathbf{W}(\mathbf{x};\delta)$ onto a neighborhood of $\mathbf{x}\mathbf{u}(s)$ in $\mathbf{W}(\mathbf{x}\mathbf{u}(s), 10\delta)$. Define

$$\mathbf{V}(\mathbf{x};\delta,1) = \bigcup\{\psi_s(\mathbf{W}(\mathbf{x};\delta))\colon 0 \le s \le 1\}.$$

Then

$$\mathbf{V}(\mathbf{x};\delta,1) = \bigcup\{\sigma_{\mathbf{y}}(1)\colon \mathbf{y} \in \mathbf{W}(\mathbf{x};\delta)\}$$

where

$$\sigma_{\mathbf{y}}(1) = \{\mathbf{y}\mathbf{u}(s)\colon 0 \le s \le \alpha(\mathbf{y}, 1)\} \subset \mathbf{y}\mathbf{U}.$$

For $\mathbf{y}, \mathbf{z} \in \mathbf{W}(\mathbf{x};\delta)$ define

$$\varphi_{\mathbf{y},\mathbf{z}}(\psi_s(\mathbf{y})) = \psi_s(\mathbf{z}) \in \psi_s(\mathbf{W}(\mathbf{x};\delta)).$$

The map $\varphi_{\mathbf{y},\mathbf{z}}$ is a diffeomorphism from $\sigma_{\mathbf{y}}(1)$ onto $\sigma_{\mathbf{z}}(1)$. Also $\varphi_{\mathbf{y},\mathbf{z}}(\mathbf{p})$ is C^∞ in $(\mathbf{y}, \mathbf{z}, \mathbf{p})$, $\mathbf{y}, \mathbf{z} \in \mathbf{W}(\mathbf{x};\delta)$, $\mathbf{p} \in \sigma_{\mathbf{y}}(1)$. This implies that given $\varepsilon > 0$ there is $\delta_0 = \delta_0(\varepsilon) > 0$ such that if $0 < \delta < \delta_0$ then

$$\left| \frac{\lambda(B)}{\lambda(\varphi_{\mathbf{y},\mathbf{z}}(B))} - 1 \right| < 0.01\varepsilon \tag{1.2}$$

for all Borel subsets $B \subset \sigma_{\mathbf{y}}(1)$ and all $\mathbf{y}, \mathbf{z} \in \mathbf{W}(\mathbf{x}; \delta)$. Here λ denotes the length measure on $\mathbf{yU}$ in which $\lambda\{\mathbf{yu}(s) \colon 0 \leq s \leq t\} = t$ for all $t \geq 0$.

For a large $t > 0$ let $\tau = \tau(t) = (\ln t)/2$ and let

$$\mathbf{W}(\mathbf{x}; \delta, t) = \mathbf{W}(\mathbf{xa}(\tau), \delta)\mathbf{a}(-\tau) = \{x\mathbf{a}(r)\mathbf{h}(b) : |r| < \delta, |b| < \delta t^{-1}\},$$
$$\mathbf{V}(\mathbf{x}; \delta, t) = \mathbf{V}(\mathbf{xa}(\tau), \delta, 1)\mathbf{a}(-\tau).$$

Also for $\mathbf{y} \in \mathbf{W}(\mathbf{x}; \delta, t)$ and $0 \leq s \leq t$ let

$$\alpha(\mathbf{y}, s) = \alpha(\mathbf{ya}(\tau), s/t)t,$$
$$\psi_s(\mathbf{y}) = \mathbf{yu}(\alpha(\mathbf{y}, s)),$$
$$\sigma_{\mathbf{y}}(t) = \{\psi_s(\mathbf{y}) : 0 \leq s \leq t\} = \{\mathbf{yu}(s) : 0 \leq s \leq \alpha(\mathbf{y}, t)\}.$$

It follows from (1.1) that

$$\psi_s(\mathbf{y}) \in \mathbf{W}(\mathbf{xu}(s); 10\delta, t)$$

for all $0 \leq s \leq t$ and all $\mathbf{y} \in \mathbf{W}(\mathbf{x}; \delta, t)$. Also

$$\lambda(\sigma_{\mathbf{y}}(t)) = \alpha(\mathbf{y}, t), \quad \lambda(\sigma_{\mathbf{x}}(t)) = t$$

and

$$\mathbf{V}(\mathbf{x}; \delta, t) = \bigcup\{\sigma_{\mathbf{y}}(t) : \mathbf{y} \in \mathbf{W}(\mathbf{x}; \delta, t)\}.$$

For $\mathbf{y}, \mathbf{z} \in \mathbf{W}(\mathbf{x}; \delta, t)$ define $\varphi_{\mathbf{y},\mathbf{z}} : \sigma_{\mathbf{y}}(t) \to \sigma_{\mathbf{z}}(t)$ by $\varphi_{\mathbf{y},\mathbf{z}}(\psi_s(\mathbf{y})) = \psi_s(\mathbf{z})$, $0 \leq s \leq t$. It follows from (1.2) that if $0 < \delta < \delta_0(\varepsilon)$ then

$$\left|\frac{\lambda(B)}{\lambda(\varphi_{\mathbf{y},\mathbf{z}}(B))} - 1\right| < 0.01\varepsilon \tag{1.3}$$

for all Borel subsets $B \subset \sigma_{\mathbf{y}}(t)$, all $\mathbf{y}, \mathbf{z} \in \mathbf{W}(\mathbf{x}; \delta, t)$ and all $t > 0$.

Now let f be a bounded uniformly continuous function on $\mathbf{G}$. Given $\varepsilon > 0$ let $\delta_f = \delta_f(\varepsilon) > 0$ be such that if $\mathbf{y}, \mathbf{z} \in \mathbf{G}$ and $d_{\mathbf{G}}(\mathbf{y}, \mathbf{z}) < \delta_f$ then

$$|f(\mathbf{y}) - f(\mathbf{z})| < 0.01\varepsilon.$$

(Here $d_{\mathbf{G}}$ denotes the left invariant metric on $\mathbf{G}$.) Define

$$\omega_f(\varepsilon) = 0.1 \min\{\delta_f(\varepsilon), \delta_0(\varepsilon C_f^{-1})\},$$
$$S_f(\mathbf{y}, t) = \frac{1}{t}\int_0^t f(\mathbf{yu}(s))ds, \qquad t > 0, \quad \mathbf{y} \in \mathbf{G},$$

where $C_f = \max\{1, |f|_\infty\}$. It follows from (1.3) that if $0 < \delta < \omega_f(\varepsilon)$ then

$$|S_f(\mathbf{y}, \alpha(\mathbf{y}, t)) - S_f(\mathbf{z}, \alpha(\mathbf{z}, t))| < 0.1\varepsilon \tag{1.4}$$

for all $\mathbf{y}, \mathbf{z} \in \mathbf{W}(\mathbf{x}; \delta, t)$ and all $t > 0$.

Now let $\mathbf{\Gamma}$ be a discrete subgroup of $\mathbf{G}$ and $\pi\colon \mathbf{G} \to \mathbf{\Gamma}\backslash\mathbf{G} = X$ the covering projection $\pi(\mathbf{g}) = \mathbf{\Gamma g}$, $\mathbf{g} \in \mathbf{G}$. The group $\mathbf{G}$ acts by right translations on X, $x \to x\mathbf{g}$, $x \in X$, $\mathbf{g} \in \mathbf{G}$.

For $x \in X$, $\mathbf{x} \in \pi^{-1}\{x\}$ let

$$W(x; \delta, t) = \pi(\mathbf{W}(\mathbf{x}; \delta, t)), \quad t > 0$$
$$V(x; \delta, t) = \pi(\mathbf{V}(\mathbf{x}; \delta, t)).$$

Now suppose that π is one-to-one on $\mathbf{W}(\mathbf{x}; \delta, t)$. For $y \in W(x; \delta, t)$ define

$$\alpha(y, s) = \alpha(\mathbf{y}, s), \quad 0 \le s \le t$$

where $\mathbf{y} = \pi^{-1}\{y\} \cap \mathbf{W}(\mathbf{x}; \delta, t)$. These notations will be used in Section 2.

For $r > 0$, $\mathbf{g} \in \mathbf{G}$ define

$$\mathbf{E}(\mathbf{g}; r) = \{\mathbf{g}\mathbf{a}(\tau)\mathbf{U}\mathbf{K}\colon r < \tau < \infty\}$$

where

$$\mathbf{K} = \left\{\mathbf{r}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} : 0 \le \theta \le 2\pi\right\}.$$

Let $\mathbf{\Gamma}$ be a nonuniform lattice in $\mathbf{G}$. Then there are $r_0 > 1$, $\mathbf{g}_1, \dots, \mathbf{g}_n \in \mathbf{G}$ and $\gamma_1, \dots, \gamma_n \in \mathbf{\Gamma}$ with $\mathbf{g}_i^{-1}\gamma_i\mathbf{g}_i \in \mathbf{U}^- = \{\mathbf{u}(s)\colon s < 0\}$ $i = 1, \dots, n$ such that if we define $\mathbf{E}_i = \mathbf{E}(\mathbf{g}_i, r_0)$, $\mathbf{\Gamma}_i = \{\gamma_i^k\colon k \in \mathbb{Z}\}$, $\tilde{\mathbf{\Gamma}} = \mathbf{\Gamma} - \{\mathbf{e}\}$ then $X - \cup\{\pi(\mathbf{E}_i)\colon i = 1, \dots, n\}$ is compact in $X = \mathbf{\Gamma}\backslash\mathbf{G}$ and

$$\begin{aligned} &\gamma_i\mathbf{E}_i = \mathbf{E}_i, \quad i = 1, \dots, n, \\ &\gamma\mathbf{E}_i \cap \mathbf{E}_i = \emptyset, \qquad \gamma \in \mathbf{\Gamma} - \mathbf{\Gamma}_i, \quad i = 1, \dots, n, \\ &\gamma\mathbf{E}_i \cap \mathbf{E}_j = \emptyset, \qquad i \ne j, \quad \gamma \in \mathbf{\Gamma}, \\ &d_{\mathbf{G}}(\mathbf{x}, \tilde{\mathbf{\Gamma}}\mathbf{x}) = d_{\mathbf{G}}(\mathbf{x}, \gamma_i\mathbf{x}), \qquad i = 1, \dots, n, \quad \mathbf{x} \in \mathbf{E}_i. \end{aligned} \tag{1.5}$$

PROPOSITION 1.1: *Let $K = X - \cup\{\pi(\mathbf{E}_i): i = 1, \dots, n\}$—a compact subset of $X = \mathbf{\Gamma}\backslash\mathbf{G}$. If $x \in X$ and $x\mathbf{U}$ is not a periodic orbit, then there exists a sequence $\tau_n \to \infty$ such that $x\mathbf{a}(\tau_n) \in K$ for all n.*

Proof: Suppose to the contrary that there exists $\tau_0 > 0$ such that $x\mathbf{a}(\tau) \notin K$ for all $\tau \geq \tau_0$. Then there exists $i \in \{1, \dots, n\}$ such that $\mathbf{x}\mathbf{a}(\tau) \in \bigcup\{\gamma\mathbf{E}_i: \gamma \in \mathbf{\Gamma}\}$ for all $\tau \geq \tau_0$, $\mathbf{x} \in \pi^{-1}\{x\}$, since $\pi(\mathbf{E}_j) \cap \pi(\mathbf{E}_k) = \emptyset$, $j \neq k$. Because $\bigcup\{\gamma\mathbf{E}_i: \gamma \in \mathbf{\Gamma}\}$ is a disjoint union, there is $\mathbf{x}_0 \in \pi^{-1}\{x\}$ such that $\mathbf{x}_0\mathbf{a}(\tau) \in \mathbf{E}_i$ for all $\tau \geq \tau_0$. But this happens if and only if $\mathbf{x}_0 \in \mathbf{g}_i\mathbf{AU}$. Hence $x\mathbf{U}$ is a periodic orbit. This gives a contradiction. ∎

2. Finite volume homogeneous spaces of SL(2, R)

A) CLASSIFICATION OF INVARIANT MEASURES AND ORBIT CLOSURES FOR HOROCYCLE FLOWS.

Proof of Theorem 2 for Lattices: Let $\mathbf{\Gamma}$ be a lattice in $\mathbf{G}$ and ν the $\mathbf{G}$-invariant Borel probability measure on $\mathbf{\Gamma}\backslash\mathbf{G} = X$. It suffices to show that if $x \in X$ and $x\mathbf{U}$ is not a closed (periodic) orbit then there is a sequence $t_n \uparrow \infty$, $n \to \infty$ such that

$$S_f(x, t_n) \to f_\nu = \int_X f d\nu, \quad n \to \infty \tag{2.1}$$

for every bounded uniformly continuous function on X.

So suppose that $x\mathbf{U}$ is not a closed orbit. By Proposition 1.1 there exist a compact subset $K \subset X$ and a sequence $\tau_n \uparrow \infty$ such that $x\mathbf{a}(\tau_n) \in K$ for all $n = 1, 2, \dots$. We claim that $t_n = e^{2\tau_n}$, $n = 1, 2, \dots$ satisfies (2.1). Indeed, let f be as above and for a given $\varepsilon > 0$ let $\omega_f(\varepsilon) = \omega_{\tilde{f}}(\varepsilon)$, where $\tilde{f}$ is the lift of f to $\mathbf{G}$. Since K is compact, there are $0 < \delta < 0.01\omega_f(\varepsilon)$ and $\eta > 0$ such that π is one-to-one on $\mathbf{W}(\mathbf{x}; \delta)$ and

$$\nu(\pi(\mathbf{V}_{0.1\varepsilon C_f^{-1}}(\mathbf{x}; \delta, 1))) > \eta$$

for all $\mathbf{x} \in \pi^{-1}(K)$, where

$$\mathbf{V}_r(\mathbf{x}; \delta, t) = \bigcup\{\psi_s(\mathbf{W}(\mathbf{x}; \delta, t)): 0 \leq s \leq r\}, \quad 0 \leq r \leq t.$$

Since the action of $\mathbf{U}$ on (X,ν) is ergodic there are $t_0 > 0$ and a subset $Y \subset X$ with $\nu(Y) > 1 - 0.1\eta$ such that

$$|S_f(y,t) - f_\nu| < 0.01\varepsilon$$

for all $y \in Y$, $t \geq t_0$. Now let $n_0 \geq 1$ be so big that $t_n \geq 100t_0$ for all $n \geq n_0$ and let $x_n = x\mathbf{a}(\tau_n) \in K$, $n \geq n_0$. Then

$$V(x;\delta,t_n) = V(x_n;\delta,1)\mathbf{a}(-\tau_n),$$
$$\nu(\pi(\mathbf{V}_{0.1\varepsilon C_f^{-1}t_n}(\mathbf{x};\delta,t_n)) > \eta,\ \mathbf{x} \in \pi^{-1}\{x\}.$$

This implies that

$$\pi(\mathbf{V}_{0.1\varepsilon C_f^{-1}t_n}(\mathbf{x};\delta,t_n)) \cap Y \neq \emptyset$$

and hence there is $y_n \in W(x;\delta,t_n)$, $n \geq n_0$ such that

$$|S_f(y_n,\alpha(y_n,t_n)) - f_\nu| < 0.5\varepsilon.$$

This gives via (1.4) that

$$|S_f(x,t_n) - f_\nu| < \varepsilon$$

for all $n \geq n_0$. This completes the proof of the Theorem. ∎

Proof of Theorem 1: It follows from the proof of Theorem 2 just given that if $x\mathbf{U}$ is not a closed orbit, then $x\mathbf{U} \cap G \neq \emptyset$ for every open subset $G \subset X$. This implies that $\overline{x\mathbf{U}} = X$. ∎

Note 2.1: The proof of Theorem 2 shows that if $X = \mathbf{\Gamma}\backslash\mathbf{G}$ is compact then $S_f(x,t) \to f_\nu$, $t \to \infty$ for **all** $x \in X$. Hence the action of $\mathbf{U}$ on X is uniquely ergodic. Other proofs of this fact are given in [F2], [B] and [EP]. Our proof of the unique ergodicity of $\mathbf{U}$ for compact $\mathbf{\Gamma}\backslash\mathbf{G}$ applies also to the uniformly parametrized horocycle flow associated with the geodesic flow on the unit tangent bundle of a compact surface of variable negative curvature. ∎

B) Equidistribution of closed horocycles.

Proof of Theorem 5: It suffices to show that

$$S_f(x,T) \to f_\nu, \quad T \to \infty$$

for every bounded uniformly continuous function f on $X = \mathbf{\Gamma}\backslash\mathbf{G}$, where $T = T(x) > 0$, $x \in P$ denotes the period of the periodic orbit $x\mathbf{U}$.

So let f and $\varepsilon > 0$ be given and let $\omega_f(\varepsilon)$ be as in the proof of Theorem 2. Let $0 < \delta < \omega_f(\varepsilon)$ be so small that π is one-to-one on $\mathbf{V}(\mathbf{z};\delta,1) - \psi_1(\mathbf{W}(\mathbf{z};\delta))$ for every $\mathbf{z} \in \mathbf{G}$ for which $\pi(\mathbf{z})\mathbf{U}$ is a periodic orbit of period 1. Let

$$\eta = \nu(\pi(\mathbf{V}_{0.01\varepsilon C_f^{-1}}(\mathbf{z};\delta,1)))$$

where $\mathbf{V}_r(\mathbf{z};\delta,1)$ is as in the proof of Theorem 2. Since the action of $\mathbf{U}$ on (X,ν) is ergodic, there are $t_0 > 1$ and $Y \subset X$ with $\nu(Y) > 1 - 0.1\eta$ such that

$$|S_f(y,t) - f_\nu| < 0.01\varepsilon$$

for all $y \in Y$, $t \geq t_0$. Arguing as in the proof of Theorem 2 we conclude that

$$|S_f(x,T) - f_\nu| < \varepsilon$$

for all $x \in P$ with $T(x) > 5t_0$. This completes the proof of the theorem. ∎

C) Uniform distribution of horocycle orbits. The reader is advised to skip this section unless he or she is particularly interested in seeing a proof of Theorem 3 which does not use Theorem 2. (Also Lemma 2.2 below is of independent interest.) A much better proof of Theorem 3 is given in Section 3 below.

Let $\mathbf{\Gamma}$ be a nonuniform lattice in $\mathbf{G}$ and let $r_0 > 1$, $\mathbf{g}_i$, $\mathbf{\Gamma}_i$, $\mathbf{E}_i$, $i = 1,\ldots,n$ be as in (1.5). If $r_0 > 0$ is sufficiently large then there is $\rho > 0$ such that

$$\{\mathbf{x} \in \mathbf{G}: d_{\mathbf{G}}(\mathbf{x},\tilde{\mathbf{\Gamma}}\mathbf{x}) \leq 3\rho\} \subset \{\gamma\mathbf{E}_i: \boldsymbol{\gamma} \in \mathbf{\Gamma},\ i = 1,\ldots,n\}$$

and if

$$\gamma\mathbf{x}\mathbf{u}(p) \in \mathbf{W}(\mathbf{x};0.1\rho)$$

for some $\mathbf{x} \in \mathbf{E}_i$, $i = 1,\ldots,n$, $0 \leq p \leq \rho$ and some $\mathbf{e} \neq \boldsymbol{\gamma} \in \mathbf{\Gamma}$ then $\boldsymbol{\gamma} \in \mathbf{\Gamma}_i$. Now we choose $0 < d < 0.1\rho$ such that

$$\mathbf{x}\mathbf{u}(s) \in \mathbf{E}_i$$

for all $|s| \leq 3d$ and all $\mathbf{x} \in \mathbf{E}_i$ with $d_{\mathbf{G}}(\mathbf{x},\tilde{\mathbf{\Gamma}}\mathbf{x}) \leq 3d$, $i = 1,\ldots,n$.

Henceforth we assume for convenience that $d = 1$. Now let $\mathbf{x} = \mathbf{g}_i\mathbf{a}(t)\mathbf{u}(s)\mathbf{r}(\theta) \in \mathbf{E}_i$ for some i and suppose that

$$\gamma\mathbf{x}\mathbf{u}(p) = \mathbf{x}\mathbf{a}(\tau)\mathbf{h}(b) \in \mathbf{W}(\mathbf{x};\delta) \tag{2.2}$$

for some $\mathbf{e} \neq \gamma \in \mathbf{\Gamma}_i$, $p \in R$ and $0 < \delta < \varepsilon_0 = 0.01$. Then

$$\mathbf{u}(q)\mathbf{r}(\theta)\mathbf{u}(p) = \mathbf{r}(\theta)\mathbf{a}(\tau)\mathbf{h}(b)$$

where $\mathbf{u}(q) = \mathbf{a}(-t)\mathbf{g}_i^{-1}\gamma\mathbf{g}_i\mathbf{a}(t) \in \mathbf{U}$, $q \neq 0$. Using this relation one can compute that

$$\begin{aligned} e^{\tau} &= 1 - q\cos\theta\sin\theta, \\ p &= -qe^{-\tau}\cos^2\theta, \\ b &= -qe^{\tau}\sin^2\theta. \end{aligned} \tag{2.3}$$

This shows that if $p \neq 0$ then $p > 0$ if and only if $q < 0$. Also $b \geq 0$, whenever $p > 0$ and

(2.4) $p(q,\theta)$ is decreasing in q for all q and all $0 \leq \theta \leq 2\pi$,

(2.5) $|\tau(q,\theta)|$ and $b = b(q,\theta)$ are decreasing for all $-\infty < q < 0$ and all θ with $\cos\theta\sin\theta \geq 1/2q$.

Relation (2.4) implies that if

$$\gamma_i\mathbf{x}\mathbf{u}(p) \in \mathbf{W}(\mathbf{x};\delta) \tag{2.6}$$

for some $0 \leq p \leq 2$ then

$$p = \min\{s \geq 0 \colon \gamma\mathbf{x}\mathbf{u}(s) \in \mathbf{W}(\mathbf{x};\delta) \text{ for some } \mathbf{e} \neq \gamma \in \mathbf{\Gamma}\}. \tag{2.7}$$

Also it follows from (2.4) and (2.5) that if (2.2) holds for some $0 \leq p \leq 2$ and $\cos\theta\sin\theta \geq 1/2q$ (in particular, when $\tau \geq 0$) then (2.6) holds and hence so does (2.7).

Now let $\mathbf{y} = \mathbf{x}\mathbf{a}(\tau)\mathbf{h}(b) \in \mathbf{W}(\mathbf{x};\delta)$ and let $\alpha(\mathbf{y},s) \geq 0$, $0 \leq s \leq 1$ be as in Section 1. Then

$$\mathbf{y}\mathbf{u}(\alpha(\mathbf{y},s)) = \mathbf{x}\mathbf{u}(s)\mathbf{a}(\tau(s))\mathbf{h}(b(s))$$

where

$$\begin{aligned} \alpha(\mathbf{y},s) &= \frac{s}{e^{2\tau} - sb}, \\ \tau(s) = \tau(\mathbf{y},s) &= \ln(e^{\tau} - sbe^{-\tau}), \\ b(s) = b(\mathbf{y},s) &= b(1 - bse^{-2\tau}). \end{aligned} \tag{2.8}$$

Now let $0 < \delta < 0.01\varepsilon_0$ be fixed and suppose that (2.2) holds for some $\mathbf{x} \in \mathbf{G}$, $0 < p \leq 1$, $\mathbf{e} \neq \boldsymbol{\gamma} \in \boldsymbol{\Gamma}$ and some $|\tau|, |b| < \delta$. Then $b \geq 0$ by (2.3). Also

$$\boldsymbol{\gamma}\mathbf{x}_s\mathbf{u}(p_s) \in \psi_s(\mathbf{W}(\mathbf{x};\delta))$$

for all $0 \leq s \leq 1$ and some $p_s \geq 0$, $p_0 = p$, where $\mathbf{x}_s = \mathbf{x}\mathbf{u}(s)$. It follows then from (2.8) that

$$(2.9) \qquad 0 \leq p_s = \frac{s}{e^{2\tau} - bs} - s + p, \qquad p_s \leq 2 \quad \text{for } 0 \leq s \leq 1.$$

Relation (2.3) shows that $p_{\bar{s}} = 0$ if and only if

$$(2.10) \qquad \bar{s} = (e^{2\tau} - e^{\tau})/b, \quad \tau \neq 0, \quad b > 0, \quad p = \bar{s}(1 - e^{-\tau}).$$

For $0 \leq s \leq 1$ define

$$\beta_s = \beta_s(\mathbf{x}, \delta) = \min\{t \geq 0 \colon \boldsymbol{\gamma}\mathbf{x}_s\mathbf{u}(t) \in \psi_s(\mathbf{W}(\mathbf{x};\delta)) \text{ for some } \mathbf{e} \neq \boldsymbol{\gamma} \in \boldsymbol{\Gamma}\}.$$

LEMMA 2.1: *Let $0 < \varepsilon < 0.1\varepsilon_0$ be given. Suppose that $\beta_0 = \beta_0(\mathbf{x}, \delta) > 0$ for some $0 < \delta < \varepsilon^5$. Then there exists $0 \leq s_0 \leq 1$ such that*

$$(2.11) \qquad \beta_s \geq \frac{1}{2}\min\{1, \beta_0\varepsilon^2\}$$

for all $s \notin [(1-\varepsilon)s_0, (1+\varepsilon)s_0]$, $0 \leq s \leq 1$.

Proof: We have

$$\boldsymbol{\gamma}_s\mathbf{x}_s\mathbf{u}(\beta_s) \in \psi_s(\mathbf{W}(\mathbf{x};\delta))$$

for all $0 \leq s \leq 1$ and some $\mathbf{e} \neq \boldsymbol{\gamma}_s \in \boldsymbol{\Gamma}$. Let

$$S = \left\{ s \in [0,1] \colon \beta_s < \frac{1}{2} \right\}.$$

It follows from the definition of ρ and d that if $s \in S$ then we can assume $\mathbf{x}_s \in \mathbf{E}_i$, $\boldsymbol{\gamma}_s \in \boldsymbol{\Gamma}_i$ for some $i \in \{1, \ldots, n\}$. Then $\mathbf{x}_s \in \mathbf{E}_i$, $0 \leq \beta_s \leq 1$, $\boldsymbol{\gamma}_s \in \boldsymbol{\Gamma}_i$ for all $0 \leq s \leq 1$. Also we can assume that

$$(2.12) \qquad \mathbf{g}_i^{-1}\boldsymbol{\gamma}_s\mathbf{g}_i \in \mathbf{U}^-$$

since this is so when $\beta_s > 0$ (by (2.3)) and if $\beta_s = 0$ and (2.12) does not hold, then we can replace $\boldsymbol{\gamma}_s$ by $\boldsymbol{\gamma}_s^{-1}$. Then

$$\boldsymbol{\gamma}_s\mathbf{x}\mathbf{u}(r_s) = \mathbf{x}\mathbf{a}(\tau_s)\mathbf{h}(b_s) \in \mathbf{W}(\mathbf{x};\delta)$$

for some $0 < \beta_0 \leq r_s \leq 1$ and all $s \in [0,1]$. Assume first that

$$\tau_s < 0 \quad \text{for all } s \in [0,1].$$

Then

$$\beta_s \geq r_s \geq \beta_0$$

for all $s \in [0,1]$ by (2.9), since $b_s \geq 0$. Then s_0 can be chosen arbitrary in (2.11). Now assume that

$$\tau_{\tilde{s}} \geq 0 \quad \text{for some } \tilde{s} \in [0,1].$$

It follows then from (2.5) that

$$\boldsymbol{\gamma}_i \mathbf{x}_s \mathbf{u}(p_s) \in \psi_s(\mathbf{W}(\mathbf{x}, \delta))$$

for all $s \in [0,1]$ and some $0 \leq p_s \leq 1$, $p_0 = p \geq \beta_0 > 0$. Then

$$\boldsymbol{\gamma}_s = \boldsymbol{\gamma}_i, \quad \beta_s = p_s$$

for all $s \in [0,1]$ by (2.7). Write $\tau_s = \tau \geq 0$, $b_s = b \geq 0$. If $b = 0$ then $\tau = 0$ and $\beta_s = \beta_0$ for all s by (2.3) and (2.9). Then s_0 can be chosen arbitrary in (2.11). So assume that $b > 0$ and let $\bar{s} > 0$ be as in (2.10). Then $p_{\bar{s}} = 0$ and $p = \bar{s}(1 - e^{-\tau})$. Now let $s_\varepsilon = \bar{s}(1 \pm \varepsilon)$. Using (2.10) and substituting s_ε instead of s into (2.9) we obtain

$$p_{s_\varepsilon} = (1 \pm \varepsilon)c(\bar{s}) + p$$

where

$$\begin{aligned} c(\bar{s}) &= \bar{s}\left[\frac{e^{-\tau}}{1 \mp \varepsilon(e^\tau - 1)} - 1\right] \\ &= \bar{s}[e^{-\tau}(1 \pm \varepsilon(e^\tau - 1) + \rho(\varepsilon, \tau)) - 1] \\ &= -p \pm \varepsilon p + \rho_1(\varepsilon, \tau, \bar{s}), \\ |\rho_1(\varepsilon, \tau, \bar{s})| &= |\bar{s}e^{-\tau}\rho(\varepsilon, \tau)| \leq 2\varepsilon^2(e^\tau - 1)^2 \bar{s} \leq \tau p. \end{aligned}$$

Then

$$p_{s_\varepsilon} = \varepsilon^2 p + (1 \pm \varepsilon)\rho_1(\varepsilon, \tau, \bar{s}) \geq \frac{1}{2}\varepsilon^2 p$$

since $0 \leq \tau < \delta < \varepsilon^5$. Set $s_0 = \min\{\bar{s}, 1 - \varepsilon\}$. Then

$$p_s \geq \frac{1}{2}\varepsilon^2 p \geq \frac{1}{2}\varepsilon^2 \beta_0$$

for all $s \in [0,(1-\varepsilon)s_0] \cup [(1+\varepsilon)s_0, 1]$, since p_s decreases in s on $[0,\bar{s}]$ and increases in s for $s > \bar{s}$ by (2.9). This completes the proof of the lemma. ∎

Now let $0 < \xi(\delta) < \delta$ be such that

$$\mathbf{W}(\mathbf{x}_s; \xi(\delta)) \subset \psi_s(\mathbf{W}(\mathbf{x}; \delta))$$

for all $0 \leq s \leq 1$. Define

$$\beta(\mathbf{x}, \delta) = \beta_0(\mathbf{x}, \delta).$$

It follows from Lemma 2.1 that if $\beta(\mathbf{x}, \delta) > 0$ for some $0 < \delta < \varepsilon^5$ then there exists $s_0 \in [0,1]$ such that

$$\beta(\mathbf{x}_s, \xi(\delta)) > \frac{1}{2} \min\{1, \varepsilon^2 \beta(\mathbf{x}, \delta)\}$$

for all $s \in [0,(1-\varepsilon)s_0] \cup [(1+\varepsilon)s_0, 1]$. Now define

$$\beta(\mathbf{x}; \delta, t) = \min\{s \geq 0 \colon \boldsymbol{\gamma}\mathbf{x}\mathbf{u}(s) \in \mathbf{W}(\mathbf{x}; \delta, t) \text{ for some } \mathbf{e} \neq \boldsymbol{\gamma} \in \boldsymbol{\Gamma}\}, \quad t \geq 1.$$

Then

$$\beta(\mathbf{x}; \delta, t) = \beta(\mathbf{x}\mathbf{a}(r), \delta)t, \quad \beta(\mathbf{x}; \delta, 1) = \beta(\mathbf{x}, \delta)$$

where $e^{2r} = t$. We get the following

Corollary 2.1: *Let $0 < \varepsilon < 0.01\varepsilon_0$ be given and let $\beta(\mathbf{x}; \delta, t) > 0$ for some $0 < \delta < \varepsilon^5$. Then there exists $s_0 \in [0,t]$ such that*

$$\beta(\mathbf{x}_s; \xi(\delta), t) \geq \frac{1}{2} \min\{t, \varepsilon^2 \beta(\mathbf{x}, \delta, t)\}$$

for all $s \in [0,(1-\varepsilon)s_0] \cup [(1+\varepsilon)s_0, t]$, where $\mathbf{x}_s = \mathbf{x}\mathbf{u}(s)$.

Lemma 2.2: *Suppose that $1 < \beta < 2t$ for some $t \geq 1$, $0 < \delta < 0.01\varepsilon_0$, where $\beta = \beta(\mathbf{x}; \delta, t)$. Then there exists $\mathbf{y}_\mathbf{x} \in \mathbf{W}(\mathbf{x}; \sqrt{10\delta}, \beta)$ such that $\pi(\sigma_{\mathbf{y}_\mathbf{x}}(\beta))$ is a closed (periodic) $\mathbf{U}$-orbit in $\boldsymbol{\Gamma}\backslash\mathbf{G}$ of length (period) $\alpha(\mathbf{y}_\mathbf{x}, \beta)$. (Here $\sigma_{\mathbf{y}_\mathbf{x}}(\beta)$ and $\alpha(\mathbf{y}_\mathbf{x}, \beta)$ are as in Section 1.)*

Proof: Let $r = \frac{1}{2}\ln\beta$ and $\mathbf{z} = \mathbf{x}\mathbf{a}(r)$. Then

$$\boldsymbol{\gamma}\mathbf{z}\mathbf{u}(1) = \mathbf{z}\mathbf{a}(\tau)\mathbf{h}(b) \in \mathbf{W}(\mathbf{z}; \delta, t/\beta)$$

for some $\mathbf{e} \neq \gamma \in \boldsymbol{\Gamma}$. It follows from the definition of ρ and d that we can assume $\mathbf{z} \in \mathbf{E}_i$, $\boldsymbol{\gamma} \in \boldsymbol{\Gamma}_i$ for some $i = 1, \ldots, n$. Then (2.3) holds with $p = 1$ and

$\frac{1}{2} \leq |q| = |pe^{\tau} + be^{-\tau}| \leq 2$, since $|\tau|, |b| \leq 2\delta$. This implies that $\sin^2\theta \leq 4\delta$ and hence

$$\text{either } |\theta| \leq \sqrt{6\delta} \text{ or } |\pi - \theta| \leq \sqrt{6\delta} \tag{2.13}$$

if δ is sufficiently small. Then $\cos\theta\sin\theta \geq \dfrac{1}{2q}$ and hence

$$\gamma_i \mathbf{z}\mathbf{u}(p') \in \mathbf{W}(\mathbf{z}; \delta, t/\beta)$$

for some $p' \geq 0$ by (2.5). It follows then from (2.4) and the definition of β that $\gamma = \gamma_i$, $p' = 1$. But $\mathbf{z} = \mathbf{z}'\mathbf{r}(\theta)$ for some $\mathbf{z}'$ with $\gamma_i\mathbf{z}'\mathbf{u}(s) = \mathbf{z}'$ for some $s > 0$. It follows then from (2.13) that there is $\mathbf{y}_\mathbf{z} \in \mathbf{W}(\mathbf{z}; \sqrt{10\delta})$ such that $\pi(\sigma_{\mathbf{y}_\mathbf{z}}(1))$ is a closed $\mathbf{U}$-orbit in $\mathbf{\Gamma}\backslash\mathbf{G}$ of length $\alpha(\mathbf{y}_\mathbf{z}, 1)$. This completes the proof of the lemma if we set $\mathbf{y}_\mathbf{x} = \mathbf{y}_\mathbf{z}\mathbf{a}(-r)$. ∎

Proof of Theorem 3: It suffices to prove that if $x \in \mathbf{\Gamma}\backslash\mathbf{G} = X$ and $x\mathbf{U}$ is not a closed orbit then

$$S_f(x,t) \to f_\nu, \text{ when } t \to \infty$$

for every bounded uniformly continuous function f on X.

So let $0 < \tilde{\varepsilon} < 0.01\varepsilon_0$ and f as above be given. Let $\varepsilon = \tilde{\varepsilon}C_f^{-1}$ and let $\omega_f(\varepsilon)$ be as in the proof of Theorem 2. Let $0 < \delta < [\min\{\varepsilon, \omega_f(\varepsilon^{10})\}]^{100}$ be so small that π is one-to-one on $\mathbf{V}(\mathbf{z}; \delta, 1) - \psi_1(\mathbf{W}(\mathbf{z}, \delta))$ for every $\mathbf{z} \in \mathbf{G}$ for which $\pi(\mathbf{z})\mathbf{U}$ is a closed $\mathbf{U}$-orbit in X of length 1. Let

$$\eta = \nu(\pi(\mathbf{V}_{\varepsilon^5}(\mathbf{z}, \xi(\delta)/2, 1))$$

where $\mathbf{V}_r(\mathbf{z}; \delta, 1)$ is as in the proof of Theorem 2 and $0 < \xi(\delta) < \delta$ as in Corollary 2.1.

Since the action of $\mathbf{U}$ on (X, ν) is ergodic, there are $l_0 > 1$ and $Y \subset X$ with $\nu(Y) > 1 - 0.1\eta$ such that

$$|S_f(y,t) - f_\nu| < \varepsilon^{10}$$

for all $y \in Y$, $t \geq l_0$. Arguing as in the proof of Theorem 2 we conclude that if $z\mathbf{U}$ is a closed orbit of length $l \geq 5l_0$ then

$$|S_f(z,l) - f_\nu| \leq 2\tilde{\varepsilon}^5. \tag{2.14}$$

Since $x\mathbf{U}$ is not a closed orbit it follows from (2.3) that there exists $t_0 \geq 10l_0/\varepsilon^4$ such that

$$\beta(\mathbf{x};\delta,t) \geq 10l_0/\varepsilon^4, \quad \mathbf{x} \in \pi^{-1}\{x\} \tag{2.15}$$

for all $t \geq t_0$. We claim that

$$|S_f(x,t) - f_\nu| < \tilde{\varepsilon} \tag{2.16}$$

for all $t \geq t_0$.

Indeed, let $t \geq t_0$. It follows from (2.15) and Corollary 2.1 that there exists $s_0 \in [0,t]$ such that

$$\beta(\mathbf{x}_s;\xi(\delta),t) \geq 10l_0$$

for all $s \in [0,(1-\varepsilon^2)s_0] \cup [(1+\varepsilon^2)s_0,(1-\varepsilon^2)t] = T$. To prove (2.16) for t it suffices to show that for each $s \in T$ there is $l_0 \leq t(s) \leq t-s$ such that

$$|S_f(x_s,t(s)) - f_\nu| < \tilde{\varepsilon}^2. \tag{2.17}$$

So let $s \in T$. If π is one-to-one on $\mathbf{V}_{t-s}(\mathbf{x}_s,\xi(\delta)/2,t)$, $\mathbf{x}_s \in \pi^{-1}\{x_s\}$ then arguing as in the proof of Theorem 2 we obtain

$$|S_f(x_s,t-s) - f_\nu| < \tilde{\varepsilon}^2.$$

We set $t(s) = t-s$ in this case. Now assume that π is not one-to-one on $\mathbf{V}_{t-s}(\mathbf{x}_s,\xi(\delta)/2,t)$. Then there are $r \in [0,t-s]$, $\mathbf{y} \in \psi_r(\mathbf{W}(\mathbf{x}_s,\xi(\delta)/2,t)$ such that

$$\gamma\mathbf{y}\mathbf{u}(p) \in \psi_r(\mathbf{W}(\mathbf{x}_s,\xi(\delta)/2,t)$$

for some $p \geq 0$ and some $\mathbf{e} \neq \gamma \in \mathbf{\Gamma}$. This implies via (2.8) that

$$\gamma\mathbf{x}_s\mathbf{u}(r') \in \mathbf{W}(\mathbf{x}_s;\xi(\delta),t)$$

for some $0 < r' < (t-s)(1+\varepsilon^8)$. This gives

$$10l_0 \leq \beta(\mathbf{x}_s;\xi(\delta),t) \leq (t-s)(1+\varepsilon^8).$$

Set $\rho(s) = \beta(\mathbf{x}_s,\xi(\delta),t)$. It follows then from Lemma 2.2 that there is $y_s \in W(x_s,\sqrt{10\xi(\delta)},\rho(s))$ such that $\sigma_{y_s}(\rho(s))$ is a closed $\mathbf{U}$-orbit of length $\alpha(y_s,\rho(s))$. Set $t(s) = \rho(s)$ if $\rho(s) \leq t-s$ and $t(s) = t-s$ if $\rho(s) > t-s$. Relation (2.17) now follows from (2.14) and our choice of δ. This completes the proof of the theorem. ∎

D) COMMENTS. Our proofs of Theorems 1, 2, 3 and 5 used the fact that **U** is a horospherical subgroup for $\mathbf{a}(\tau)$, $\tau > 0$, i.e. $\mathbf{U} = \{\mathbf{g} \in \mathbf{G}\colon \mathbf{a}(-n\tau)\mathbf{g}\mathbf{a}(n\tau) \to \mathbf{e}, n \to \infty\}$. This is not necessarily true if **U** is a one-parameter unipotent subgroup of a general Lie group **G**. Because of this, our proofs can not be extended to the general case. In Section 3 and Section 4 (which handles arbitrary discrete Γ) we give proofs of Theorems 3 and 2, which **can** be extended to the general case. This was done in [R, 1-4]. Note that an analog of Theorem 2 for general horospherical **U** is given in [R1, Theorem 4] (see also [D1] and [V]).

3. A better proof of Theorem 3

In this section we give a better proof of Theorem 3, which incorporates in a simple form some of the ideas used to prove Theorem C (see [R4, Proof of Theorem 2.1]). The proof uses Theorem 2.

Let Γ be a lattice in $\mathbf{G} = \mathrm{SL}(2, R)$. We will need the following theorem.

THEOREM 3.1: *Given $\varepsilon > 0$ there is a compact $K(\varepsilon) \subset X = \Gamma\backslash\mathbf{G}$ such that if $\mathbf{U} = \{\mathbf{u}(s)\colon s \in R\}$ is a one-parameter unipotent subgroup of $\mathbf{G}$, $z \in X$ and $z\mathbf{U}$ is not a periodic orbit then*

$$\int_0^t \chi_{K(\varepsilon)}(z\mathbf{u}(s))ds \geq (1-\varepsilon)t \tag{3.1}$$

for all $t \geq t_0$ and some $t_0 = t_0(z, \mathbf{U}, \varepsilon) > 0$, where χ_K denotes the characteristic function of K.

A general version of this theorem was proved in [D2, Theorem 3.5] and used in [R4, Proof of Theorem 2.1].

Let $\mathbf{U} = \{\mathbf{u}(t) = \exp tu\colon t \in R\}$, $u \in \mathfrak{G}$ be a one-parameter unipotent subgroup of **G** and $v \in \mathfrak{G}$. Then $|\ \mathrm{Ad}_{\mathbf{u}(s)}(v)|\ ^2$ is a polynomial in s of degree ≤ 4, where $\mathrm{Ad}_{\mathbf{g}}(v) = \frac{d}{dt}(\mathbf{g}^{-1}(\exp tv)\mathbf{g})|_{t=0}$, $\mathbf{g} \in \mathbf{G}$. This fact plays an important role in the proof of Theorem 3.1. Indeed, we prove the following

LEMMA 3.1: *Let $\mathcal{P}(k)$ be the set of all real (or complex) polynomials of degree $\leq k$. Then given $\varepsilon > 0$ and $\theta > 0$ there is $0 < \delta = \delta(\varepsilon, \theta, k) < \theta$ such that if $P \in \mathcal{P}(k)$ and*

$$\max\{|P(s)|\colon 0 \leq s \leq t\} = \theta \tag{3.2}$$

for some $t > 0$ then

$$\lambda\{s \in [0,t]: |P(s)| \geq \delta\} > (1-\varepsilon)t$$

where λ denotes the length measure on R with $\lambda([0,t]) = t$.

Proof: Using a standard scaling argument it suffices to assume that $t = 1$ in (3.2). Let $C([0,1])$ denote the Banach space of all continuous functions on $[0,1]$ with the supremum norm. For $f \in C([0,1])$, $\alpha \geq 0$ and $\varepsilon > 0$ define

$$A(f,\alpha) = \{x \in [0,1]: |f(x)| \geq \alpha\},$$
$$\varphi_\varepsilon(f) = \sup\{\alpha \geq 0: \lambda(A(f,\alpha)) \geq 1-\varepsilon\}.$$

It is easy to see that $|\varphi_\varepsilon(f) - \varphi_\varepsilon(g)| \leq |\, f-g|$ for all $f,g \in C([0,1])$ and hence $\varphi_\varepsilon(f)$ is continuous on $C([0,1])$. Now let

$$\mathcal{P}_\theta = \{P \in \mathcal{P}(k): |\,P|_{[0,1]} = \theta\}.$$

Then $\mathcal{P}_\theta$ is a closed and bounded subset of the finite dimensional subspace $\mathcal{P}(k) \subset C([0,1])$. Hence $\mathcal{P}_\theta$ is compact and hence $\varphi_\varepsilon(P) \geq \delta_0$ for all $P \in \mathcal{P}_\theta$ and some $\delta_0 = \delta_0(\varepsilon,\theta,k) > 0$. This completes the proof. ∎

Now let $\tilde{\mathbf{\Gamma}} = \mathbf{\Gamma} - \{\mathbf{e}\}$ and for $\mathbf{x} \in \mathbf{G}$ let

$$\Delta(\mathbf{x}) = d_{\mathbf{G}}(\mathbf{x}, \tilde{\mathbf{\Gamma}}\mathbf{x}).$$

Also let $\mathbf{E}_i, \boldsymbol{\gamma}_i$, $i = 1,\ldots,n$ and $\rho > 0$ be as on page 10. For $0 < r < \rho$, $i = 1,\ldots,n$ define

$$\mathbf{E}_i(r) = \{\mathbf{x} \in \mathbf{E}_i: \Delta(\mathbf{x}) \leq r\} \subset \mathbf{E}_i(\rho) \subset \mathbf{E}_i.$$

Now suppose that $\mathbf{U}$ is a one-parameter subgroup of $\mathbf{G}$ and

$$d_{\mathbf{G}}(\mathbf{x}\mathbf{u}(s), \boldsymbol{\gamma}_i\mathbf{x}\mathbf{u}(s)) \leq \theta$$

for all $0 \leq s \leq t$, some $t > 0$, $\mathbf{x} \in \mathbf{E}_i$, $i \in \{1,\ldots,n\}$ and $0 < \theta < 0.5\rho$. Then

$$\text{(3.3)} \qquad \mathbf{x}\mathbf{u}(s) \in \mathbf{E}_i(\theta), \quad d_{\mathbf{G}}(\mathbf{x}\mathbf{u}(s), \boldsymbol{\gamma}_i\mathbf{x}\mathbf{u}(s)) = \Delta(\mathbf{x}\mathbf{u}(s))$$

for all $0 \leq s \leq t$ by the definition of $\mathbf{E}_i$ and $\mathbf{E}_i(\theta)$.

It follows from the definition of $\mathbf{E}_i$ that if $\mathbf{x} \in \mathbf{G}$ and $\Delta(\mathbf{x}) \leq \rho$ then there is a unique $i \in \{1, \ldots, n\}$ and $\tilde{\gamma}_{\mathbf{x}} \in \mathbf{\Gamma}$ such that $\tilde{\gamma}_{\mathbf{x}}\mathbf{x} \in \mathbf{E}_i$. Then defining $\gamma_{\mathbf{x}} = \tilde{\gamma}_{\mathbf{x}}^{-1}\gamma_i\tilde{\gamma}_{\mathbf{x}}$ we get

$$\Delta(\tilde{\gamma}_{\mathbf{x}}\mathbf{x}) = d_{\mathbf{G}}(\tilde{\gamma}_{\mathbf{x}}\mathbf{x}, \gamma_i\tilde{\gamma}_{\mathbf{x}}\mathbf{x}) = d_{\mathbf{G}}(\mathbf{x}, \gamma_{\mathbf{x}}\mathbf{x}) = \Delta(\mathbf{x}).$$

This implies via (3.3) that if

$$d_{\mathbf{G}}(\mathbf{x}\mathbf{u}(s), \gamma_{\mathbf{x}}\mathbf{x}\mathbf{u}(s)) \leq \theta$$

for all $0 \leq s \leq t$ and some $t > 0$, $\mathbf{x} \in \mathbf{G}$, $0 < \theta < 0.5\rho$ then

$$d_{\mathbf{G}}(\mathbf{x}\mathbf{u}(s), \gamma_{\mathbf{x}}\mathbf{x}\mathbf{u}(s)) = \Delta(\mathbf{x}\mathbf{u}(s)) \tag{3.4}$$

for all $0 \leq s \leq t$.

Proof of Theorem 3.1: Let $\varepsilon > 0$ be given and let $0 < \theta < \min\{1, 0.5\rho\}$ be so small that if $d_{\mathbf{G}}(\mathbf{x}, \mathbf{y}) \leq 2\theta$ for some $\mathbf{x}, \mathbf{y} \in \mathbf{G}$ then $\mathbf{y} = \mathbf{x} \exp v$ for some $v \in \mathfrak{G}$ with $|\,v| = d_{\mathbf{G}}(\mathbf{x}, \mathbf{y})$. Let $0 < \delta^2 = \delta(0.1\varepsilon, \theta^2/4, 4) < \theta^2/4$ be as in Lemma 3.1 for $k = 4$. Let

$$K(\varepsilon) = \{x \in X \colon \Delta(\mathbf{x}) \geq \delta,\ \mathbf{x} \in \pi^{-1}\{x\}\}$$

—a compact subset of X. Now let $\mathbf{U}$ be a one-parameter unipotent subgroup of $\mathbf{G}$ and $z \in X$. Suppose that $z\mathbf{U}$ is not a periodic orbit. If $z\mathbf{u}(s) \in K(\varepsilon)$ for all $s \geq 0$ then we are done. Otherwise there exists $s_0 \geq 0$ such that $z\mathbf{u}(s_0) \notin K(\varepsilon)$. Then there is $i \in \{1, \ldots, n\}$ and $\mathbf{y} \in \mathbf{E}_i$ such that $\pi(\mathbf{y}) = z\mathbf{u}(s_0)$ and

$$d_{\mathbf{G}}(\mathbf{y}, \gamma_i\mathbf{y}) < \delta.$$

Then $\gamma_i\mathbf{y} = \mathbf{y}\exp v$ for some $v \in \mathfrak{G}$ with $|\,v| < \delta$ and $\exp v \notin \mathbf{U}$ since $z\mathbf{U}$ is not periodic. Hence there is $\tau > 0$ such that

$$\begin{aligned} d_{\mathbf{G}}(\mathbf{y}\mathbf{u}(\tau), \gamma_i\mathbf{y}\mathbf{u}(\tau)) &= \theta \\ d_{\mathbf{G}}(\mathbf{y}\mathbf{u}(s), \gamma_i\mathbf{y}\mathbf{u}(s)) &\leq \theta \end{aligned} \tag{3.5}$$

for all $0 \leq s \leq \tau$. Hence

$$\Delta(\mathbf{w}) = \theta \quad \text{where } \mathbf{w} = \mathbf{y}\mathbf{u}(\tau) \tag{3.6}$$

by (3.3) and (3.5). Now let $t > 1$. Define

$$F(\mathbf{w}, t) = \{s \in [0, t] \colon \Delta(\mathbf{w}\mathbf{u}(s)) < \delta\}$$

and let $s_1 = \sup F(\mathbf{w}, t)$, $\mathbf{w}_1 = \mathbf{w}\mathbf{u}(s_1)$. Then

$$d_G(\mathbf{w}_1, \gamma_{\mathbf{w}_1}\mathbf{w}_1) \leq \delta$$

where $\gamma_{\mathbf{w}_1}$ is as in (3.4). It follows from (3.6) that

$$d_G(\mathbf{w}, \gamma_{\mathbf{w}_1}\mathbf{w}) > 0.5\theta.$$

Hence there is $0 < r_1 < s_1$ such that

$$d_G(\mathbf{w}_1\mathbf{u}(r_1 - s_1),\ \gamma_{\mathbf{w}_1}\mathbf{w}_1\mathbf{u}(r_1 - s_1)) = 0.5\theta,$$
$$d_G(\mathbf{w}_1\mathbf{u}(-s), \gamma_{\mathbf{w}_1}\mathbf{w}_1\mathbf{u}(-s)) \leq 0.5\theta,$$

for all $s \in [0, s_1 - r_1]$. It follows then from (3.4) and Lemma 3.1 that

$$\lambda\{s \in [0, s_1 - r_1]\colon \mathbf{w}_1\mathbf{u}(-s) \in \mathbf{K}(\varepsilon)\} > (1 - 0.1\varepsilon)(s_1 - r_1)$$

where $\mathbf{K}(\varepsilon) = \pi^{-1}(K(\varepsilon))$. Hence

$$\lambda\{s \in I_1\colon \mathbf{w}\mathbf{u}(s) \in \mathbf{K}(\varepsilon)\} \geq (1 - \varepsilon)\lambda(I_1)$$

where $I_1 = [r_1, t]$. By repeated application of this argument we obtain $s_1 > s_2 > \cdots > s_m > 0$, $r_1 > r_2 > \cdots > r_m = 0$, $s_{k+1} < r_k < s_k$, $k = 1, \ldots, m-1$ such that

$$s_k = \sup(F(\mathbf{w}, t) \cap [0, r_{k-1}]), \quad r_0 = t, \quad k = 1, \ldots, m$$

and

$$\lambda\{s \in I_k\colon \mathbf{w}\mathbf{u}(s) \in \mathbf{K}(\varepsilon)\} \geq (1 - 0.1\varepsilon)\lambda(I_k),$$
$$I_k = [r_k, r_{k-1}], \quad [0, t] = \bigcup_{k=1}^{m} I_k.$$

This implies (3.1) if we set $t_0 = 100(s_0 + \tau)/\varepsilon$. This completes the proof of the theorem. ∎

Now let $x \in X$ be fixed. Let $C_0(X)$ denote the Banach space of all real continuous functions on X vanishing at infinity with the supremum norm and let $C_0^*(X)$ denote the dual of $C_0(X)$. For $t > 0$ define $T_{x,t} \in C_0^*(X)$ by

$$T_{x,t}(f) = \frac{1}{t}\int_0^t f(x\mathbf{u}(s))ds, \quad f \in C_0(X).$$

Then $|T_{x,t}| \leq 1$. Let $\mathcal{T}_x$ denote the set of all limit points in the weak $*$-topology on $C_0^*(X)$ of the set $\{T_{x,t}: t > 0\}$ when $t \uparrow \infty$. For each $T \in \mathcal{T}_x$ there is a unique Borel measure μ_T on X such that

$$T(f) = \int_X f d\mu_T, \quad f \in C_0(X).$$

It is clear that $\mu_T(X) \leq 1$ and μ_T is U-invariant. Write $M(x, \mathbf{U}) = \{\mu_T: T \in \mathcal{T}_x\}$. For each $\mu \in M(x,\mathbf{U})$ there is a subsequence $t_n = t_n(\mu) \uparrow \infty$, $n \to \infty$ such that

$$T(t_n, f) = T_{x,t_n}(f) \to \int_X f d\mu$$

for all $f \in C_0(X)$.

The proof of the following lemma uses standard arguments and can be found in [R4, Proposition 1.2]. In this lemma A_δ denotes the δ-neighborhood of $A \subset X$ in X.

Lemma 3.2: *Let $\mu \in M(x,\mathbf{U})$ and let $0 < t_n = t_n(\mu) \uparrow \infty$ be as above. Let $K \subset X$ be a compact subset of X. Then, given $\varepsilon > 0$ there is $\delta_0 = \delta_0(\varepsilon, K) > 0$ such that*

$$\mu(K) \leq \liminf_{n\to\infty} T(t_n, \chi_{K_\delta}) \leq \limsup_{n\to\infty} T(t_n, \chi_{K_\delta}) \leq \mu(K) + \varepsilon$$

for all $0 < \delta < \delta_0$.

This lemma implies via Theorem 3.1 that $\mu(X) = 1$ for all $\mu \in M(x, \mathbf{U})$.

Let $\mu \in M(x,\mathbf{U})$ and let $Y_\mu \subset \overline{x\mathbf{U}}$ be the support of μ. Let $\{(C(y), \mu_{C(y)}): y \in Y'_\mu\}$ be the ergodic decomposition of the action of $\mathbf{U}$ on (Y_μ, μ), $Y'_\mu \subset Y_\mu$, $\mu(Y'_\mu) = 1$. Let $\bar{C}(y)$ denote the support of $\mu_{C(y)}$ and let $\xi_\mu = \{\bar{C}(y): y \in Y'_\mu\}$. It follows from Theorem 2 that if $\bar{C}(y) \in \xi_\mu$ then either $\bar{C}(y) = X$ and $\mu_{C(y)}$ is G-invariant or $\bar{C}(y) = y\mathbf{U}$ is a periodic orbit of $\mathbf{U}$ with $\mu_{C(y)}$ being the normalized length measure on $y\mathbf{U}$. Let $\zeta_\mu = \{C \in \xi: C \text{ is a periodic orbit of } \mathbf{U}\}$.

Proof of Theorem 3: It suffices to prove that if $\beta_\mu = \mu(\cup\{C: C \in \zeta_\mu\}) > 0$ for some $\mu \in M(x, \mathbf{U})$ then $x\mathbf{U}$ is a periodic orbit.

Indeed, let $\beta = \beta_\mu > 0$ for some $\mu \in M(x,\mathbf{U})$. Let $K = K(0.01\beta)$ be as in Theorem 3.1 and let D be a compact subset of $\cup\{C: C \in \zeta_\mu\}$ with $\mu(D) > 0.9\beta$. It follows then from (1.1) that there exists $\tau > 0$ such that $D\mathbf{a}(\tau) \subset X - K$. Lemma 3.2 implies that

$$\liminf_{n\to\infty} T_{x,t_n}(\chi_{D_\delta}) \geq 0.9\beta$$

for all small $\delta > 0$. Hence

$$\liminf T_{z,t_n e^{-2\tau}}(\chi_{(D\mathbf{a}(\tau))_\delta}) \geq 0.9\beta$$

for all small $\delta > 0$, where $z = x\mathbf{a}(\tau)$. This implies that relation (3.1) does not hold for z and $\varepsilon = 0.01\beta$. Then $z\mathbf{U}$ must be periodic by Theorem 3.1. Hence so is $x\mathbf{U}$, since $\mathbf{a}(\tau)$ normalizes $\mathbf{U}$. This proves our theorem. ∎

4. Arbitrary homogeneous spaces of SL$(2, R)$

A) Classification of invariant measures for horocycle flows. In this section we shall prove Theorem 2. Thus we assume that Γ is an arbitrary discrete subgroup of $\mathbf{G}$. Since $\mathbf{G}$ is unimodular, $\Gamma\backslash\mathbf{G}$ carries a σ-finite $\mathbf{G}$-invariant Borel measure ν.

The central role in the proof of Theorem 2 is played by a dynamical property of $\mathbf{U}$, called the R-property which was first introduced in [R5]. To state it we turn again to $\mathbf{W}(\mathbf{x};\delta)$ defined in Section 1 for a small $0 < \delta < 0.1$. It follows from (2.8) that if $\mathbf{y} = \mathbf{x}\mathbf{a}(\tau)\mathbf{h}(b) \in \mathbf{W}(\mathbf{x};\delta)$ and $bs < e^{2\tau}$ for some $s \in R$ then

$$\mathbf{y}\mathbf{u}(\alpha(\mathbf{y}, s)) = \mathbf{x}\mathbf{u}(s)\mathbf{a}(\tau(\mathbf{y}, s))\mathbf{h}(b(\mathbf{y}, s))$$

where $\tau(\mathbf{y}, s)$, $b(\mathbf{y}, s)$ and $\alpha(\mathbf{y}, s)$ are as in (2.8). Relations (2.8) imply the following statement.

The R-property for horocycle flows. There exist $0 < \eta < 1$ and $C > 1$ such that if

$$\max\{|\tau(\mathbf{y}, s)|: 0 \leq s \leq t\} = |\tau(\mathbf{y}, t)| = \theta$$

for some $t > 1$, $\mathbf{y} \in \mathbf{W}(\mathbf{x}, \delta)$, $10\delta < \theta < 0.5$ then

$$\frac{\theta}{2} \leq |\tau(\mathbf{y}, s)| \leq \theta, \quad |b(\mathbf{y}, s)| \leq \frac{C\theta}{t} \tag{4.1}$$

for all $s \in [(1-\eta)t, t]$.

This property has been extended to simply connected unipotent subgroups of general Lie groups $\mathbf{G}$ in [R1]. It plays a crucial role in the proof of Theorem B.

Now let μ be an ergodic $\mathbf{U}$-invariant Borel probability measure on $X = \Gamma\backslash\mathbf{G}$ and let $\Lambda = \Lambda(\mu) = \{\mathbf{g} \in \mathbf{G} : \text{the action of } \mathbf{g} \text{ on } X \text{ preserves } \mu\}$. Then $\Lambda(\mu)$ is a closed subgroup of $\mathbf{G}$ and $\mathbf{U} \subset \Lambda(\mu)$. Let

$$\mathbf{Q} = \{\mathbf{a}(r)\mathbf{u}(s): r, s \in R\}.$$

Then $\mathbf{Q}$ normalizes $\mathbf{U}$.

LEMMA 4.1: *Suppose that* $\mathbf{Q}-\Lambda \neq \emptyset$. *Then there exists* $Y \subset X$, $\mu(Y)=1$ *such that* $Y \cap Y\mathbf{q} = \emptyset$ *for all* $\mathbf{q} \in \mathbf{Q}-\Lambda$.

Proof: First let us show that for every $\mathbf{q} \in \mathbf{Q}-\Lambda$ there is $X_{\mathbf{q}} \subset X$, $\mu(X_{\mathbf{q}})=1$ and $\varepsilon(\mathbf{q})>0$ such that

$$X_{\mathbf{q}} \cap X_{\mathbf{q}}\mathbf{g} = \emptyset \tag{4.2}$$

for all $\mathbf{g} \in \mathbf{q}\mathbf{Q}_{\varepsilon(\mathbf{q})}(\mathbf{e}) = \mathbf{Q}_{\varepsilon(\mathbf{q})}(\mathbf{q})$, where $\mathbf{Q}_{\varepsilon}(\mathbf{e})$ denotes the ε-ball at $\mathbf{e}$ in $\mathbf{Q}$.

For $\mathbf{q} \in \mathbf{Q}-\Lambda$ define

$$\mu_{\mathbf{q}}(E) = \mu(E\mathbf{q})$$

for every Borel subset $E \subset X$. It is clear that $\mu_{\mathbf{q}}$ is an ergodic $\mathbf{U}$-invariant measure on X, since $\mathbf{q}$ normalizes $\mathbf{U}$. Also $\mu_{\mathbf{q}} \neq \mu$, since $\mathbf{q} \notin \Lambda$. Then $\mu_{\mathbf{q}}$ is singular with respect to μ and hence there exists $E_{\mathbf{q}} \subset X$ such that

$$\mu(E_{\mathbf{q}}) = 1 \text{ and } \mu_{\mathbf{q}}(E_{\mathbf{q}}) = \mu(E_{\mathbf{q}}\mathbf{q}) = 0.$$

Let $E'_{\mathbf{q}} = E_{\mathbf{q}} - E_{\mathbf{q}}\mathbf{q}$. Then $\mu(E'_{\mathbf{q}}) = 1$ and

$$E'_{\mathbf{q}} \cap E'_{\mathbf{q}}\mathbf{q} = \emptyset.$$

Now let K be a compact subset of $E'_{\mathbf{q}}$ with $\mu(K) > 0.99$. Then there is $\varepsilon = \varepsilon(\mathbf{q}) > 0$ such that

$$d_X(K, K\mathbf{q}) \geq \varepsilon. \tag{4.3}$$

Since $\mathbf{U}$ acts ergodically on (X,μ) there is $X_{\mathbf{q}} \subset X$, $\mu(X_{\mathbf{q}})=1$ such that

$$S_{\chi_K}(x,t) = \frac{1}{t}\int_0^t \chi_K(x\mathbf{u}(s))ds \to \mu(K), \quad t \to \infty \tag{4.4}$$

for all $x \in X_{\mathbf{q}}$. We claim that (4.2) holds for all $\mathbf{g} \in \mathbf{Q}_{\varepsilon(\mathbf{q})}(\mathbf{q})$. Indeed, suppose to the contrary that

$$X_{\mathbf{q}} \cap X_{\mathbf{q}}\mathbf{g} \neq \emptyset$$

for some $\mathbf{g} \in \mathbf{Q}_{\varepsilon(\mathbf{q})}(\mathbf{q})$. Then there is $x \in X_{\mathbf{q}}$ such that $x = y\mathbf{g}$ for some $y \in X_{\mathbf{q}}$. We have $\mathbf{g} = \mathbf{a}(\tau)\mathbf{u}(r)$ for some $\tau, r \in R$. Also $y\mathbf{u}(s)\mathbf{g} = x\mathbf{u}(se^{-2\tau})$ for all $s \in R$. It follows from (4.4) that there is $t > 1$ such that

$$S_{\chi_K}(y,t) \geq 0.9,$$
$$S_{\chi_K}(x, e^{-2\tau}t) \geq 0.9.$$

This implies that there is $0 \le s \le t$ such that

$$y\mathbf{u}(s) = z \in K \text{ and } z\mathbf{g} \in K.$$

But $z\mathbf{g} = z\mathbf{q}\mathbf{p}$ for some $\mathbf{p} \in \mathbf{Q}_{\varepsilon(\mathbf{q})}(\mathbf{e})$. Hence

$$d_X(z\mathbf{g}, z\mathbf{q}) < \varepsilon(\mathbf{q})$$

in contradiction with (4.3). This proves (4.2).

We have

$$\mathbf{Q} - \mathbf{\Lambda} \subset \bigcup_{i=1}^{\infty} \mathbf{Q}_{\varepsilon(\mathbf{q}_i)}(\mathbf{q}_i)$$

for some $\mathbf{q}_i \in \mathbf{Q} - \mathbf{\Lambda}$, $i = 1, 2, \ldots$. Let

$$Y = \bigcap_{i=1}^{\infty} X_{\mathbf{q}_i}.$$

Then $\mu(Y) = 1$ and

$$Y \cap Y\mathbf{g} = \emptyset$$

for all $\mathbf{g} \in \mathbf{Q} - \mathbf{\Lambda}$ by (4.2). This completes the proof of the lemma. ∎

A more general version of Lemma 4.1 is proved in [R1, Theorem 2.2].

THEOREM 4.1: *Suppose* $\mathbf{A} \not\subset \mathbf{\Lambda}(\mu)$. *Then* μ *is supported on a closed orbit of* $\mathbf{U}$.

Proof: Since $\mathbf{A} \not\subset \mathbf{\Lambda}$ and $\mathbf{\Lambda}$ is a closed subgroup of $\mathbf{G}$ there is $0 < \theta < 0.1$ such that

$$\mathbf{a}(\tau) \notin \mathbf{\Lambda}$$

for all $0 < |\tau| \le \theta$. We can assume that $\theta < \delta_0(0.1)$ where $\delta_0(\varepsilon)$ is as in (1.2).

Thus $\mathbf{a}(\tau) \in \mathbf{Q} - \mathbf{\Lambda}$ for all $0 < |\tau| \le \theta$. Let $Y \subset X$, $\mu(Y) = 1$ be as in Lemma 4.1 and let $0 < \eta < 1$ be as in the R-property. Then there are a compact $K \subset Y$, $\mu(K) > 1 - 10^{-3}\eta$ and $\delta = \delta(K) > 0$ such that

$$d_X(K, K\mathbf{a}(\tau)) \ge \delta \tag{4.5}$$

for all $\theta/2 \le |\tau| \le \theta$. Since the action of $\mathbf{U}$ on (X, μ) is ergodic, there are $F \subset X$, $\mu(F) > 0$ and $t_0 \ge 1$ such that

$$S_{\chi_K}(x, t) \ge 1 - 10^{-2}\eta \tag{4.6}$$

for all $x \in F$, $t \geq t_0$.

Now let $0 < \xi < 0.01\theta$ be so small that if $\max\{|\tau(\mathbf{y}, s)|: 0 \leq s \leq t\} = |\tau(\mathbf{y}, t)| = \theta$ for some $\mathbf{y} \in \mathbf{W}(\mathbf{x}; \xi)$, $\mathbf{x} \in \mathbf{G}$ and $t \geq 1$ then

$$t \geq 10t_0 \text{ and } C\theta/t \leq 0.01\delta.$$

Here $C \geq 1$ is as in (4.1). We claim that if $x, y \in F$ and $d_X(x, y) < \xi$ then

$$y \in Q(x; \xi) = \{x\mathbf{a}(r)\mathbf{u}(s): |r|, |s| < \xi\}.$$

Indeed, suppose to the contrary that $y \notin Q(x; \xi)$. We can assume without loss of generality that $y = x\mathbf{a}(\tau)\mathbf{h}(b) \in W(x; \xi)$. Then $b \neq 0$. It follows then from (2.8) that there is $t = t(y) > 0$ such that $|\tau(y, t)| = \theta = \max\{|\tau(y, s)|: 0 \leq s \leq t\}$. Then $t \geq t_0$ and $\alpha(y, t) \geq t_0$ by our choice of ξ and θ. It follows then from (4.6) that

$$S_{X_K}(x, t) \geq 1 - 0.01\eta$$
$$S_{X_K}(y, \alpha(y, t)) \geq 1 - 0.01\eta.$$

This implies by our choice of θ that there is $s \in [(1 - \eta)t, t]$ such that

$$x\mathbf{u}(s) \in K \text{ and } x\mathbf{u}(s)\mathbf{a}(\tau(y, s))\mathbf{h}(b(y, s)) \in K.$$

Then

$$\frac{\theta}{2} \leq |\tau(y, s)| \leq \theta, \quad |b(y, s)| \leq C\theta/t \leq 0.1\delta$$

by the R-property. This gives

$$d_X(K, K\mathbf{a}(\tau(y, s))) \leq 0.1\delta$$

in contradiction with (4.5).

Now let $x \in F \cap Y$ be such that $\mu(F \cap O_\varepsilon(x)) > 0$ for all $\varepsilon > 0$, where $O_\varepsilon(x)$ denotes the ε-ball at x in X. We have just shown that

$$\mu(Q(x; \xi) \cap Y) > 0.$$

This implies via Lemma 4.1 that

$$Q(x; \xi) \cap Y \subset x\mathbf{U}$$

since $x \in Y$. Hence $\mu(x\mathbf{U}) = 1$, since $\mathbf{U}$ acts ergodically on (X, μ). This completes the proof of the theorem. ∎

Now we shall prove the following

THEOREM 4.2: *Suppose* $\mathbf{A} \subset \Lambda(\mu)$. *Then* Γ *is a lattice and* μ *is* G*-invariant.*

To prove this theorem we need the following lemma.

LEMMA 4.2: *Suppose* $\mathbf{A} \subset \Lambda(\mu)$. *Then the action of* $\mathbf{A}$ *on* (X, μ) *is mixing.*

Proof: It suffices to show that

$$\int_X \varphi(x) f(x\mathbf{a}(-\tau)) d\mu \to 0, \text{ when } \tau \to \infty$$

for any two bounded uniformly continuous functions φ and f on X with $f_\mu = \int_X f d\mu = 0$.

So let $\varepsilon > 0$ be given and let $0 < \delta < 1$ be such that

$$|\varphi(x) - \varphi(z)| < \varepsilon \tag{4.7}$$

for all $x, z \in X$, $d_X(x, z) < \delta$. Since the action of $\mathbf{U}$ on (X, μ) is ergodic there are $t_0 > 1$ and $Y \subset X$, $\mu(Y) > 1 - \varepsilon$ such that

$$|S_f(y, t)| < \varepsilon \tag{4.8}$$

for all $y \in Y$, $t \geq t_0$.

Now let $\tau_0 > 0$ be such that $e^{-2\tau_0} t_0 = \delta$ and let $\tau \geq \tau_0$. Write

$$Y_\tau = Y\mathbf{a}(\tau), \quad \mu(Y_\tau) = \mu(Y) > 1 - \varepsilon$$

since $\mathbf{a}(\tau) \in \Lambda(\mu)$. We have using (4.7)

$$\begin{aligned}
I(\tau) &= \int_X \varphi(x) f(x\mathbf{a}(-\tau)) d\mu \\
&= \frac{1}{\delta} \int_0^\delta \left(\int_X \varphi(x\mathbf{u}(s)) f(x\mathbf{u}(s)\mathbf{a}(-\tau)) d\mu \right) ds \\
&= \int_X \left(\frac{1}{\delta} \int_0^\delta \varphi(x\mathbf{u}(s)) f(x\mathbf{u}(s)\mathbf{a}(-\tau)) ds \right) d\mu \\
&= \int_X \varphi(x) \left[\frac{1}{\delta} \int_0^\delta f(x\mathbf{a}(-\tau)\mathbf{u}(e^{2\tau}s)) ds \right] d\mu + \varepsilon_1 \\
&= \int_X \varphi(x) \left[\frac{1}{s_\tau} \int_0^{s_\tau} f(x\mathbf{a}(-\tau)\mathbf{u}(s)) ds \right] d\mu + \varepsilon_1 \\
&= \int_{Y_\tau} \varphi(y) S_f(y\mathbf{a}(-\tau), s_\tau) d\mu + \varepsilon_1 + \varepsilon_2
\end{aligned}$$

where $s_\tau = \delta e^{2\tau} \geq t_0$, $y\mathbf{a}(-\tau) \in Y$ whenever $y \in Y_\tau$ and $|\varepsilon_1|, |\varepsilon_2| \leq C_1\varepsilon$ for some $C_1 > 0$. This gives via (4.8)

$$|I(\tau)| \leq C\varepsilon$$

for all $\tau \geq \tau_0$ and some $C > 0$. This completes the proof of the lemma. ∎

A more general version of this lemma is proved in [R1, Theorem 5].

Thus we assume that $\mathbf{A} \subset \Lambda(\mu)$. Then μ is preserved by the action of $\mathbf{Q} = \{\mathbf{a}(\tau)\mathbf{u}(s) \colon \tau, s \in R\}$ on X.

Now let $x \in X$ and $H(x;\delta) = \{x\mathbf{h}(s) \colon |s| \leq \delta\}$. If $0 < \delta < 0.1$ is sufficiently small then for each $y \in Q(x;\delta)$ and each $z \in H(x;\delta)$ the intersection $H(y;10\delta) \cap Q(z;10\delta)$ consists of exactly one point $p = p(y,z)$. Define

$$H(y) = H(p) = \{p(y,v) \colon v \in H(x;\delta)\},$$
$$Q(z) = Q(p) = \{p(w,z) \colon w \in Q(x;\delta)\},$$
$$B_\delta(x) = \bigcup_{y \in Q(x;\delta)} H(y).$$

We have

$$B_\delta(x) = \bigcup_{q \in H(p)} Q(q) = \bigcup_{r \in Q(p)} H(r)$$

for all $p \in B_\delta(x)$. The set $B_\delta(x)$ is similar to the set $\cup\{\psi_s(W(x;\delta)) \colon |s| \leq \delta\}$ discussed in Section 1. We can assume without loss of generality that $\mu(B_{\delta/2}(x)) > 0$ and π is one-to-one on the 10δ-ball $\mathbf{O}_{10\delta}(\mathbf{x})$ at $\mathbf{x} \in \pi^{-1}\{x\}$ in $\mathbf{G}$.

Define

$$\Omega = \cup\{B_\delta(x)\mathbf{a}^k \colon k \in \mathbb{Z}\}.$$

Then $\mu(\Omega) = 1$, since the action of $\mathbf{a}$ on (X,μ) is ergodic. Also the action of $\mathbf{a}$ on (Ω,ν) is measure preserving. Let $\bar{\nu}$ be the Borel measure on X defined by $\bar{\nu}(D) = \nu(D \cap \Omega)$ for every Borel subset $D \subset X$.

LEMMA 4.3: 1) $\nu(\Omega) < \infty$; 2) $\mu = \bar{\nu}/\nu(\Omega)$.

Proof: Let f be a continuous function on X with compact support. Since the action of $\mathbf{a}$ on (X,μ) is ergodic, there is a subset $C_f \subset B_\delta(x)$, $\mu(C_f) = \mu(B_\delta(x))$ such that if $y \in C_f$ then

$$S_{f,n}(y) = \sum_{i=0}^{n-1} f(y\mathbf{a}^{-i})/n \to f_\mu = \int_X f\,d\mu, \quad n \to \infty. \tag{4.9}$$

Let $\tilde{C}_f \subset B_\delta(x)$, $\mu(\tilde{C}_f) = \mu(B_\delta(x))$ be such that if $z \in \tilde{C}_f$ then

$$\lambda(C_f \cap Q(z))/\lambda(Q(z)) = 1$$

where λ denotes a $\mathbf{Q}$-invariant measure on $z\mathbf{Q}$. Pick $\tilde{z} \in \tilde{C}_f$ and define

$$B_f = \cup\{H(y)\colon y \in C_f \cap Q(\tilde{z})\} \subset B_\delta(x),$$
$$\Omega_f = \cup\{B_f \mathbf{a}^k\colon k \in \mathbb{Z}\} \subset \Omega.$$

We have $\nu(B_f) = \nu(B_\delta(x))$ and $\nu(\Omega_f) = \nu(\Omega)$. Now let $z \in B_f$. Then $z \in H(y)$ for some $y \in C_f$. We have

$$d_X(z\mathbf{a}^{-n}, y\mathbf{a}^{-n}) \to 0, \quad n \to \infty.$$

This and (4.9) imply that

$$S_{f,n}(z) \to f_\mu, \quad n \to \infty$$

for all $z \in B_f$, since f is uniformly continuous. Also

$$S_{f,n}(\omega) \to f_\mu, \quad n \to \infty \tag{4.10}$$

for all $\omega \in \Omega_f$. Now let f be nonnegative with $f_\mu \neq 0$. It follows then from the Fatou's lemma that

$$f_\mu \nu(\Omega) = \int_{\Omega_f} f_\mu d\nu \leq \lim_{n\to\infty} \int_{\Omega_f} S_{f,n} d\nu = \int_\Omega f d\nu < \infty.$$

This proves that $\nu(\Omega) < \infty$. Now we use (4.10) and the Lebesgue Dominated Convergence Theorem to get

$$f_{\bar{\nu}} = \int_\Omega f d\nu = \int_\Omega S_{f,n} d\nu \to \int_\Omega f_\mu d\nu = f_\mu \nu(\Omega)$$

for every continuous function f on X with compact support. This proves that $\mu = \bar{\nu}/\nu(\Omega)$. ∎

Proof of Theorem 4.2: In view of Lemma 4.3 it remains to prove that $\nu = \bar{\nu}$. To do so it suffices to show that for every $p \in X$

$$\nu(O_{0.1\delta}(p) - \Omega) = 0$$

where $O_\gamma(p) = p\mathbf{O}_\gamma(\mathbf{e})$. Define

$$\bar{\Omega} = \{\omega \in \Omega : \omega \mathbf{a}^{-n} \in B_{\delta/2}(x) \text{ for intinitely many } n \in \mathbb{Z}^+\}.$$

We have $\mu(\bar{\Omega}) = 1$ and $\nu(\bar{\Omega}) = \nu(\Omega)$, since $\mu = \hat{\nu} = \bar{\nu}/\nu(\Omega)$ and the action of $\mathbf{a}$ on (Ω, μ) is ergodic. If $\omega \in \bar{\Omega}$ then $H(\omega; 10\delta)\mathbf{a}^{-n} \subset H(y)$ for some $n \in \mathbb{Z}^+$ and some $y \in B_\delta(x)$. This implies that

$$H(\omega, 10\delta) \subset \Omega$$

for all $\omega \in \bar{\Omega}$, since Ω is $\mathbf{a}$-invariant. In fact, $\omega\mathbf{H} \subset \Omega$ for all $\omega \in \bar{\Omega}$. Now let

$$\hat{\Omega} = \{\omega \in \Omega : \lambda(\bar{\Omega} \cap Q(\omega; 10\delta))/\lambda(Q(\omega, 10\delta)) = 1\}.$$

We have

$$\nu(\hat{\Omega}) = \nu(\Omega) \tag{4.11}$$

since $\hat{\nu} = \mu$ is $\mathbf{Q}$-invariant. It follows now from the definition of $\hat{\Omega}$ that if $\omega \in \hat{\Omega}$ then

$$\nu(B_\delta(\omega) \cap \Omega) = \nu(B_\delta(\omega)). \tag{4.12}$$

This implies via (4.11) that

$$\nu(B_\delta(\omega) \cap \hat{\Omega}) = \nu(B_\delta(\omega))$$

for all $\omega \in \hat{\Omega}$. Now let $p \in X$. Then we can find $x = \omega_1, \dots \omega_n$ such that $\omega_i \in B_\delta(\omega_{i-1}) \cap \hat{\Omega}$, $i = 2, \dots, n$ and $O_{0.1\delta}(p) \subset B_\delta(\omega_n)$. This implies via (4.12) that

$$\nu(O_{0.1\delta}(p) - \Omega) = 0$$

and proves that $\nu = \bar{\nu}$. ∎

A similar proof for a more general case is given in [R2, Section 7].

Proof of Theorem 2: The theorem follows from Theorems 4.1 and 4.2. ∎

B) ORBIT CLOSURES FOR HOROCYCLE FLOWS. In this section we prove Theorem 4. Thus we assume that $\mathbf{\Gamma}$ is a discrete subgroup of $\mathbf{G}$ and $\mathbf{\Gamma}$ is not a lattice. Suppose that $x \in \mathbf{\Gamma}\backslash\mathbf{G} = X$ and $\overline{x\mathbf{U}}$ is compact in X. Let $M(x, \mathbf{U})$ be as in section 3. Then $\mu(X) = 1$ for all $\mu \in M(x, \mathbf{U})$.

Proof of Theorem 4: Let $\mu \in M(x, \mathbf{U})$ and let $Y_\mu \subset \overline{x\mathbf{U}}$ denote the support of μ. By Theorem 2 there is $y \in Y_\mu$ such that $y\mathbf{U}$ is a periodic orbit. Since $\overline{x\mathbf{U}}$ is compact, there are $r > 1$ and $\varepsilon > 0$ such that

$$d_X(y\mathbf{U}\mathbf{a}(r), x\mathbf{U}) > \varepsilon. \tag{4.13}$$

Now suppose to the contrary that $x\mathbf{U}$ is not periodic. Since $y\mathbf{U} \subset \overline{x\mathbf{U}}$ there are $t > 0$ and $z \in y\mathbf{U}$ such that

$$p = x\mathbf{u}(t) = z\mathbf{a}(\tau)\mathbf{h}(b) \in W(z; \delta)$$

for some $|\tau|, |b| < \delta$ and $b \neq 0$, where $\delta > 0$ is chosen so small that $\delta < 0.01\varepsilon e^{-r}$. It follows then from (2.8) that if $e^\tau - sbe^{-\tau} = e^r$ then

$$\tau(y, s) = r, \quad |b(y, s)| \leq 0.1\varepsilon.$$

Then

$$d_X(p\mathbf{u}(\alpha(y, s)), z\mathbf{u}(s)\mathbf{a}(r)) < 0.1\varepsilon$$

in contradiction with (4.13), since $z\mathbf{u}(s) \in y\mathbf{U}$. This completes the proof of the theorem. ∎

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